Smile Asymptotics II: Models with Known Moment Generating Function

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Abstract

In a recent article the authors obtained a formula which relates explicitly the tail of risk neutral returns with the wing behavior of the Black Scholes implied volatility smile. In situations where precise tail asymptotics are unknown but a moment generating function is available we first establish, under easy-to-check conditions, tail asymptoics on logarithmic scale as soft applications of standard Tauberian theorems. Such asymptotics are enough to make the tail-wing formula work and we so obtain a version of Lee's moment formula with the novel guarantee that there is indeed a limiting slope when plotting implied variance against log-strike. We apply these results to time-changed Lévy models and the Heston model. In particular, the term-structure of the wings can be analytically understood.

1 Introduction

Consider a random variable X whose moment generating function (mgf) M is known in closed form, but whose density f (if it exists) and distribution function F are, even asymptotically, unknown. For a large class of distributions used for modelling (risk-neutral) returns in finance, M is finite only on part of the real line. Let us define $\bar{F} \equiv 1 - F$ and r^* as the least upper bound of all real r for which $M(r) \equiv E[e^{rX}] < \infty$ and assume $r^* \in (0, \infty)$. An easy Chebyshev argument gives

$$\limsup_{x\to\infty} \frac{-\log \bar{F}(x)}{x} = r^*, \tag{1}$$

but counter-examples show that the stronger statement

$$-\log \bar{F}(x) \sim r^* x \quad \text{as } x \to \infty \tag{2}$$

may not be true[1]. However, we do expect (2) to be true if the (right) tail of the distribution is reasonably behaved. Our interest in such distributions stems

[1] We use the standard notation $g(x) \sim h(x) \equiv \lim g(x)/h(x) = 1$ as $x \to \infty$.

from the fact that the crude tail asymptotics (2) and the mild integrability condition $p^* = r^* - 1 > 0$ are enough, via the tail-wing formula [4], to assert existence of a limiting slope of Black Scholes implied variance V^2 as function of log-strike k. Indeed, in standard notation, reviewed in section 4, one has

$$\lim_{k\to\infty} V^2(k)/k = 2 - 4\left(\sqrt{(p^*)^2 + p^*} - p^*\right). \tag{3}$$

Similarly, if $q^* \equiv \sup\left\{q \in \mathbb{R} : M\left(-q\right) \equiv E[e^{-qX}] < \infty\right\} \in (0,\infty)$ and the (left) tail is reasonably behaved one expects $\log F(-x) \sim -q^* x$ as $x \to \infty$ in which case the tail wing formula gives

$$\lim_{k\to\infty} V^2\left(-k\right)/k = 2 - 4\left(\sqrt{(q^*)^2 + q^*} - q^*\right). \tag{4}$$

It was already pointed out in [4] that the tail-wing formulae sharpen Lee's celebrated moment formulae [9, 8]. In the present context, this amounts to having a lim instead of a lim sup[2]. It must be noted that the tail-wing formula requires some knowledge of the tails whereas the moment formula is conveniently applicable by looking at the mgf (to obtain the critical values r^* and $-q^*$).

In this paper we develop criteria, checkable by looking *a little closer* at the mgf (near r^* and $-q^*$), which will guarantee that (3) resp. (4) hold. In view of the tail-wing formula the problem is reduced to obtain criteria for (2) resp. its left-sided analogue. The proofs rely on Tauberian theorems and, as one expects, the monograph [5] is our splendid source.

The criteria are then fine-tuned to the fashionable class of time-changed Lévy models [11, 6] and checked explicitly for the examples of Variance Gamma under Gamma-OU clock and Normal Inverse Gaussian with CIR clock. We also check the criteria for the Heston model. In fact, it appears to us that most (if not all) sensible models for stock returns with known mgf and $p^*, q^* \in (0,\infty)$ satisfy one of our criteria so that (3) and (4) will hold.

Finally, we present some numerical results. The asymptotic regime becomes visible for remarkably low log-strikes which underlines the practical value of moment - and tail-wing formulae.

2 Background in Regular Variation

2.1 Asymptotic inversion

If $f = f\left(x\right)$ is defined and locally bounded on $[X,\infty)$, and tends to ∞ as $x \to \infty$ then the generalized inverse

$$f^{\leftarrow}\left(x\right) := \inf\left\{y \in [X,\infty) : f\left(y\right) > x\right\}$$

[2] Remark that, at least when $p^* > 0$, the moment formula is in fact recovered from the tail-wing formula and (1).

is defined on $[f(X), \infty)$ and is monotone increasing to ∞. This applies in particular to $f \in R_\alpha$ with $\alpha > 0$ and Thm 1.5.12 in [5] asserts that $f^{\leftarrow} \in R_{1/\alpha}$ and

$$f\left(f^{\leftarrow}(x)\right) \sim f^{\leftarrow}(f(x)) \sim x \text{ as } x \to \infty.$$

Given f one can often compute $f^{\leftarrow}$ (up the asymptotic equivalence) in terms of the *Bruijn conjugate* of slowly varying functions (Prop. 1.5.15, Section 5.2. and Appendix 5 in [5]).

2.2 Smooth Variation

A positive function g defined in some neighbourhood of ∞ *varies smoothly with index* α, $g \in SR_\alpha$, iff $h(x) := \log(g(e^x))$ is C^∞ and

$$h'(x) \to \alpha, \;\; h^{(n)}(x) \to 0 \text{ for } n = 2, 3, \ldots \text{ as } x \to \infty.$$

Theorem 1 (Smooth Variation Theorem, Thm 1.8.2 in [5]) *If $f \in R_\alpha$ then there exist $f_i \in SR_\alpha$, $i = 1, 2$, with $f_1 \sim f_2$ and $f_1 \leq f \leq f_2$ on some neighbourhood of ∞.*

When $\alpha > 0$ we can assume that f_1 and f_2 are strictly increasing in some neighbourhood of ∞. In fact, we have

Proposition 2 *Let $\alpha > 0$ and $g \in SR_\alpha$. Then g is strictly increasing in some neighbourhood of ∞ and $g' \in SR_{\alpha-1}$.*

Proof. By definition of SR_α,

$$\frac{\partial}{\partial x} \log(g(e^x)) = \frac{1}{g(e^x)} g'(e^x) e^x \to \alpha > 0 \text{ as } x \to \infty.$$

This shows that, in some neighbourhood of ∞, g' is strictly positive which implies that g is strictly increasing. From Prop 1.8.1 in [5], $g' = |g'| \in SR_{\alpha-1}$. ■

Remark 3 *In the situation of the last Proposition we have $\lim_{x\to\infty} g(x) = \infty$ and hence, in some neighbourhood of ∞ , g has a genuine inverse g^{-1} which coincides with the generalized inverse $g^{\leftarrow}$.*

2.3 Exponential Tauberian Theory

Theorem 4 (Kohlbecker's Theorem, Thm 4.12.1 and Cor 4.12.6 in [5]) *Let U be a non-decreasing right-continuous function on $\mathbb{R}$ with $U(x) = 0$ for all $x < 0$. Set*

$$N(\lambda) := \int_{[0,\infty)} e^{-x/\lambda} dU(x), \;\; \lambda > 0.$$

Let $\alpha > 1$ and $\chi \in R_{\alpha/(\alpha-1)}$. Then

$$\log N(\lambda) \sim (\alpha - 1)\, \chi(\lambda) / \lambda \textit{ as } \lambda \to \infty$$

iff

$$\log \mu [0,x] \sim \alpha x / \chi^{\leftarrow}(x) \ \ as\ x \to \infty.$$

Theorem 5 (Karamata's Tauberian Theorem, Thm 1.7.1 in [5]) *Let U be a non-decreasing right-continuous function on $\mathbb{R}$ with $U(x) = 0$ for all $x < 0$. If $l \in R_0$ and $c \geq 0, \rho \geq 0$, the following are equivalent:*

$$U(x) \sim c x^{\rho} l(x) / \Gamma(1+\rho) \ \ as\ x \to \infty$$

$$\hat{U}(s) \equiv \int_0^{\infty} e^{-sx} dU(x) \sim c s^{-\rho} l(1/s) \ \ as\ s \to 0+.$$

(When $c = 0$ the asymptotic relations are interpreted in the sense that $U(x) = o(x^{\rho} l(x))$ and similar for $\hat{U}$.)

Theorem 6 (Bingham's Lemma, Thm 4.12.10 in [5]) *Let $f \in R_{\alpha}$ with $\alpha > 0$ such that that e^{-f} is locally integrable at $+\infty$. Then*

$$-\log \int_x^{\infty} e^{-f(y)} dy \sim f(x).$$

3 Moment generating functions and log-tails

Let F be a finite Borel measure on $\mathbb{R}$, identified with its (bounded, non-decreasing, right-continuous) distributions function, $F(x) \equiv F((-\infty, x])$. Its mgf is defined as

$$M(s) := \int e^{sx} dF(x).$$

We define the critical exponents q^* and r^* via

$$-q^* \equiv \inf\{s : M(s) < \infty\}, r^* \equiv \sup\{s : M(s) < \infty\}$$

and make the **standing assumption** that

$$r^*, q^* \in (0, \infty).$$

In this section, we develop criteria which will imply the asymptotic relations

$$\log F((-\infty, -x]) \sim -q^* x, \ \log F((x, \infty)) \sim -r^* x \text{ as } x \to \infty.$$

The assumption in the following Criterion I is simply that some derivative of the mgf (at the critical exponent) blows up in a regularly varying way.

Theorem 7 (Criterion I) *Let F be a bounded non-decreasing right-continuous function on $\mathbb{R}$ and define $M = M(s)$, q^* and r^* as above.*
(i) If for some $n \geq 0$, $M^{(n)}(-q^* + s) \sim s^{-\rho} l_1(1/s)$ for some $\rho > 0$, $l_1 \in R_0$ as $s \to 0+$ then

$$\log F((-\infty, -x]) \sim -q^* x$$

(ii) If for some $n \geq 0$, $M^{(n)}(r^* - s) \sim s^{-\rho} l_1(1/s)$ for some $\rho > 0$, $l_1 \in R_0$ as $s \to 0+$ then

$$\log F((x, \infty)) \sim -r^* x.$$

Proof. Let us focus on case (ii), noting that case (i) is similar. We first discuss $n = 0$. The idea is an Escher-type change of measure followed by an application of Karamata's Tauberian Theorem. We define a new measure U on $[0, \infty)$ by a change-of-measure designed to get rid of the exponential decay,

$$dU(x) := \exp(r^* x)\, dF(x).$$

We identify U with its non-decreasing right-continuous distribution function $x \mapsto U([0, x])$. The Laplace transform of U is given by

$$\hat{U}(s) = \int_0^\infty e^{-sx} dU(x) = \int_0^\infty e^{(r^*-s)x} dF(x) = M(r^* - s) - \int_{-\infty}^0 e^{(r^*-s)x} dF(x)$$

so that

$$\left|\hat{U}(s) - M(r^* - s)\right| \leq \int_{-\infty}^0 e^{(r^*-s)x} dF(x) \leq F(0) - F(-\infty) \leq 2\|F\|_\infty < \infty.$$

Since $M(r^* - s)$ goes to ∞ as $s \to 0+$ and we see that $\hat{U}(s) \sim M(r^* - s)$ so that $\hat{U} \in R_\rho$ as $s \to 0$. Hence, there exists $l \in R_0$ so that $\hat{U}(s) = (1/s)^\rho\, l(1/s)$ and Karamata's Tauberian theorem tells us that $U \in R_\rho$, namely

$$U(x) \sim x^\rho l(x) / \Gamma(1 + \rho) \equiv x^\rho l'(x) \text{ as } x \to \infty$$

where $l' \in R_0$. Going back to the right-tail of F, we have for $x \geq 0$

$$F((x, \infty)) = \int_{(x,\infty)} dF(y) = \int_{(x,\infty)} \exp(-r^* y)\, dU(y).$$

We first assume that $U \in SR_\rho$. Under this assumption U is smooth with derivative $u = U' \in SR_{\rho-1}$ and we can write

$$u(y) = y^{\rho-1} l''(y) \text{ with } l'' \in R_0.$$

Then

$$\begin{aligned} F((x, \infty)) &= \int_{(x,\infty)} \exp(-r^* y)\, y^{\rho-1} l''(y)\, dy \\ &= \int_{(x,\infty)} \exp\left[-r^* y + (\rho - 1)\log y + \log l''(y)\right] dy. \end{aligned}$$

Since $-\left[-r^* y + (\rho - 1)\log y + \log l''(y)\right] \sim r^* y \in R_1$ as $y \to \infty$ we can use Bingham's lemma to obtain

$$-\log F((x, \infty)) = -\log \int_{(x,\infty)} \exp\left[-r^* y + (\rho - 1)\log y + \log l''(y)\right] dU(y) \sim r^* y. \tag{5}$$

We now deal with the general case of non-decreasing $U \in R_\rho$. From the Smooth Variation Theorem and Proposition 2 we can find $U_-, U_+ \in SR_\rho$, strictly increasing in a neighbourhood of ∞, so that

$$U_- \leq U \leq U_+ \text{ and } U_- \sim U \sim U_+.$$

Below we use the change of variable $z = U(y)$ and $w = U_+^{-1}(z)$. Noting that $U_+^{-1} \leq U^{\leftarrow} \leq U_-^{-1}$ and using change-of-variable formulae, as found in [10, p7-9] for instance, we have

$$\begin{aligned}
F((x,\infty)) &= \int_{(x,\infty)} \exp(-r^* y)\, dU(y) \\
&= \int_{(U(x),\infty)} \exp(-r^* U^{\leftarrow}(z))\, dz \\
&\leq \int_{(U(x),\infty)} \exp\left(-r^* U_+^{-1}(z)\right) dz \\
&= \int_{(U_+^{-1}(U(x)),\infty)} \exp(-r^* w)\, dU_+(w).
\end{aligned}$$

Similar to the derivation of (5), Bingham's lemma leads to

$$-\log \int_{(U_+^{-1}(U(x)),\infty)} \exp(-r^* w)\, dU_+(w) \sim r^* U_+^{-1}(U(x)).$$

Noting that U_+^{-1} is non-decreasing, $U_+^{-1}(U(x)) \leq U_+^{-1}(U_+(x)) = x$ so that[3]

$$-\log F([x,\infty)) \lesssim r^* x$$

The same argument gives the lower bound $-\log F((x,\infty)) \gtrsim r^* x$ and we conclude that

$$-\log F((x,\infty)) \sim r^* x.$$

We now show how $n > 0$ follows from $n = 0$. Define V on $[0,\infty)$ by

$$dV(x) := x^n dF(x).$$

Clearly, V induces a non-decreasing, right continuous distribution on $\mathbb{R}$, $V(x) := V([0,x])$ for $x \geq 0$ and $V(x) \equiv 0$ for $x < 0$. The distribution function $V(x)$ is also bounded since

$$\int_0^\infty x^n dF(x) < \infty$$

which follows a forteriori from the standing assumption of exponential moments. We will write $\bar{V}(x)$ for $V(x,\infty)$.

[3] By $g \lesssim h$ we mean $\limsup f(x)/g(x) \leq 1$ as $x \to \infty$.

Note that V has a mgf $M_V(s)$, finite at least for $s \in (0, r^*)$, given by

$$M_V(s) \equiv \int e^{sx} dV(x) = \int_0^\infty x^n e^{sx} dF$$
$$= \int x^n e^{sx} dF + C = M^{(n)}(s) + C$$

where[4]

$$0 \leq C \equiv -\int_{-\infty}^0 x^n e^{sx} dF \leq \int_{-\infty}^0 |x|^n dF < \infty.$$

By assumption, $M^{(n)}$ is regularly varying with index ρ at r^* and it follows that, as $s \to 0+$,

$$M_V(r^* - s) = M^{(n)}(r^* - s) + O(1) \sim s^{-\rho} l_1(1/s).$$

We now use the "$n = 0$" result on the distribution function V resp. its mgf M_V and obtain

$$-\log V([x, \infty)) \equiv -\log \bar{V}(x) \sim r^* x \in R_1$$

Assume first that $-\log \bar{V}(x) \in SR_1$. Then V has a density $V' \equiv v$ and

$$v(x) = \partial_x (V(\infty) - \bar{V}(x)) = -\bar{V}(x) \partial_x \left(\log \bar{V}(x)\right) \sim r^* \bar{V}(x) \text{ as } x \to \infty$$

since functions in SR_1 are stable under differentiation in the sense that $\partial_x \left(-\log \bar{V}(x)\right) \sim \partial_x (r^* x) = r^*$. In particular, we have $\log v(x) \sim \log \bar{V}(x) \sim -r^* x$. After these preparations we can write

$$F((x, \infty)) = \int_{(x,\infty)} dF(y)$$
$$= \int_{(x,\infty)} \frac{1}{y^n} v(y) dy$$
$$= \int_{(x,\infty)} \exp[\log v(y) - n \log y] dy$$

and Bingham's lemma implies that $\log F((x, \infty)) \sim -r^* x$. The general case of $\log \bar{V}(x) \in R_1$ follows by a smooth variation and comparison argument as earlier. ■

The next criterion deals with exponential blow-up of M at its critical values.

Theorem 8 (Criterion II) *Let F, M, q^*, r^* be as above.*
(i) If $\log M(-q^* + s) \sim s^{-\rho} l_1(1/s)$ for some $\rho > 0$, $l_1 \in R_0$ as $s \to 0+$ then

$$\log F((-\infty, -x]) \sim -q^* x$$

(ii) If $\log M(r^* - s) \sim s^{-\rho} l_1(1/s)$ for some $\rho > 0$, $l_1 \in R_0$ as $s \to 0+$ then

$$\log F((x, \infty)) \sim -r^* x.$$

[4] One could do without the assumption $\int_{-\infty}^0 |x| dF$ (which follows a forteriori from the standing assumption $q^* > 0$). Finiteness of F on $(-\infty, 0)$ is enough.

Proof. As for Criterion I, the idea is an Escher-type change of measure followed by a suitable Tauberian theorem; in the present case we need Kohlbecker's Theorem. Let us focus on case (ii), noting that case (i) is similar. A new measure U on $[0,\infty)$ is defined by

$$dU(x) := \exp(r^* x)\, dF(x).$$

We identify U with its non-decreasing right-continuous distribution function $x \mapsto U([0,x])$ and define the transform

$$N(\lambda) = \int_0^\infty e^{-x/\lambda} dU(x) = \int_0^\infty e^{(r^*-1/\lambda)x} dF(x) = M(r^* - 1/\lambda) - \int_{-\infty}^0 e^{(r^*-1/\lambda)x} dF(x)$$

so that

$$|N(\lambda) - M(r^* - 1/\lambda)| \le \int_{-\infty}^0 e^{(r^*-1/\lambda)x} dF(x) \le F(0) - F(-\infty) \le 2\|F\|_\infty < \infty.$$

Thus,

$$N(\lambda) = M(r^* - 1/\lambda) + O(1) \text{ as } \lambda \to \infty$$

and, in particular, since $\lim_{\lambda\to\infty} \log M(r^* - 1/\lambda) = \lim_{\lambda\to\infty} M(r^* - 1/\lambda) = \infty$ from the assumption (ii) we see that

$$\log N(\lambda) \sim \log M(r^* - 1/\lambda) \quad \sim \lambda^\rho l_1(\lambda) \in R_\rho \text{ as } \lambda \to \infty.$$

Define $\alpha \in (1,\infty)$ as the unique solution to $\rho + 1 = \alpha/(\alpha - 1)$ and note

$$\chi(\lambda) := \frac{\lambda}{(\alpha-1)} \log N(\lambda) \in R_{\rho+1} = R_{\alpha/(\alpha-1)}.$$

Using that $\chi^{\leftarrow} \in R_{(\alpha-1)/\alpha} = R_{1-1/\alpha}$, Kohlbecker's Tauberian Theorem tells us that

$$\log U([0,x]) \equiv \log U(x) \sim \alpha x / \chi^{\leftarrow}(x) \in R_{1/\alpha} \text{ as } x \to \infty.$$

In particular, there exists $l \in R_0$ so that $\log U(x) = \alpha x^{1/\alpha} l(x)$. We first assume that $\log U \in SR_{1/\alpha}$. Then U has a density $u(.) \in SR_{1/\alpha - 1}$ and

$$u(x) = U(x)\, \partial_x (\log U(x)) \sim U(x)\, x^{1/\alpha - 1} l(x).$$

In particular,

$$\log u(x) \sim \log U(x) \in R_{1/\alpha} \text{ as } x \to \infty.$$

Now, $y \mapsto r^* y \in R_1$ dominates $R_{1/\alpha}$ (since $1/\alpha < 1$) in the sense that

$$r^* y - \log u(y) \sim r^* y.$$

Thus, from

$$\begin{aligned} F((x,\infty)) &= \int_{(x,\infty)} dF(y) = \int_{[x,\infty)} \exp(-r^* y)\, u(y)\, dy \\ &= \int_{(x,\infty)} \exp[-r^* y + \log u(y)] \end{aligned}$$

and Bingham's lemma we deduce that

$$-\log F\left((x,\infty)\right)\sim r^{*}x.$$

The general case, $\log U\in R_{1/\alpha}$, is handled via smooth variation as earlier. Namely, we can find smooth minorizing and majorizing functions for $\log U$, say G_{-} and G_{+}. After defining $U_{\pm}=\exp G_{\pm}$ we have

$$\log U_{-}\sim\log U\sim\log U_{+}\text{ and }U_{-}\leq U\leq U_{+}.$$

Then, exactly as in the last step of the proof of Criterion I,

$$F\left((x,\infty)\right)=\int_{(x,\infty)}\exp\left(-r^{*}y\right)dU\left(y\right)\leq\int_{\left(U_{+}^{-1}(U(x)),\infty\right)}\exp\left(-r^{*}w\right)dU_{+}\left(w\right)$$

and from Bingham's lemma,

$$-\log F\left((x,\infty)\right)\lesssim r^{*}U_{+}^{-1}\left(U\left(x\right)\right)\sim r^{*}x.$$

Similarly, $-\log F\left((x,\infty)\right)\gtrsim r^{*}x$ and the proof is finished. ■

4 Application to Smile Asymptotics

We start with a few recalls to settle the notation. The normalized price of a Black-Scholes call with log-strike k is given by

$$c_{BS}\left(k,\sigma\right)=\Phi\left(d_{1}\right)-e^{k}\Phi\left(d_{2}\right)$$

with $d_{1,2}\left(k\right)=-k/\sigma\pm\sigma/2$. If one models risk-neutral returns with a distribution function F, the implied volatility is the (unique) value $V\left(k\right)$ so that

$$c_{BS}\left(k,V\left(k\right)\right)=\int_{k}^{\infty}\left(e^{x}-e^{k}\right)dF\left(x\right)=:c\left(k\right).$$

Set $\psi\left[x\right]\equiv 2-4\left(\sqrt{x^{2}+x}-x\right)$ and recall $\bar{F}\equiv 1-F$. The following is a special case of the tail-wing formula [4].

Theorem 9 *Assume that* $-\log F\left(-k\right)/k\sim q^{*}$ *for some* $q^{*}\in(0,\infty)$. *Then*

$$V(-k)^{2}/k\sim\psi\left[-\log F\left(-k\right)/k\right]\sim\psi\left(q^{*}\right).$$

Similarly, assume that $-\log\bar{F}\left(k\right)/k\sim p^{*}+1$ *for some* $p^{*}\in(0,\infty)$. *Then*

$$V(k)^{2}/k\sim\psi\left[-1-\log\bar{F}\left(k\right)/k\right]\sim\psi\left(p^{*}\right).$$

As earlier, let $M\left(s\right)=\int\exp\left(sx\right)dF\left(x\right)$ denote the mgf of risk-neutral returns and now *define* the critical exponents $r^{*}$and $-q^{*}$ exactly as in the beginning of the last section 3. Combining the results therein with Theorem 9 we obtain

Theorem 10 *If $q^* \in (0, \infty)$ and M satisfies part (i) of Criteria I or II then*

$$V(-k)^2/k \sim \psi(q^*) \quad \text{as } k \to \infty.$$

Similarly, if $r^* \equiv p^* + 1 \in (1, \infty)$ and M satisfies part (ii) of Criteria I or II then

$$V(k)^2/k \sim \psi(p^*) \quad \text{as } k \to \infty.$$

5 First Examples

The examples discussed in this section model risk-neutral log-price by Lévy processes and there is no loss of generality to focus on unit time.[5]

5.1 Criterion I with n=0: the Variance Gamma Model

The Variance Gamma model $VG = VG(m, g, C)$ has mgf

$$M(s) = \left(\frac{gm}{gm + (m-g)s - s^2}\right)^C = \left(\frac{gm}{(m-s)(s+g)}\right)^C.$$

The critical exponents are obviously given by $r^* = m$ and $q^* = g$. Focusing on the first, we have

$$M(r^* - s) \sim \left(\frac{gm}{m+g}\right)^C s^{-C} \text{ as } s \to 0+$$

which shows the Criterion I is satisfied with $n = 0$. Theorem 10 now identifies the asymptotic slope of the implied variance to be $\psi(r^* - 1) = \psi(m - 1)$. Similarly, the left slope is seen to be $\psi(q^*) = \psi(g)$. We remark that [1] contains tail estimates for VG which lead, via the tail-wing formula, to the same result.

5.2 Criterion I with n>0: the Normal Inverse Gaussian Model

The Normal Inverse Gaussian Model $NIG = NIG(\alpha, \beta, \mu, \delta)$ has mgf given by

$$M(s) = \exp\left\{\delta\left\{\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+s)^2}\right\} + \mu s\right\}.$$

By looking at the endpoints of the strip of analyticity the critical exponents are immediately seen to be $r^* = \alpha - \beta$, $q^* = \alpha + \beta$ and we focus again on the first. While $M(s)$ converges to the finite constant $M(r^*)$ as $s \to r^*-$ we have

$$M'(s)/M(s) = (2\delta(\beta+s)[\alpha^2 - (\beta+s)^2]^{-1/2} + \mu)$$
$$\text{and } M'(r^* - s) \sim 2\delta\alpha\sqrt{2\alpha}s^{-1/2}M(r^*) \text{ as } s \to 0+.$$

[5]In fact, Lévy models that satisfy one of our criteria have no term structure of implied variance slopes.

We see that Criterion I is satisfied with $n = 1$ and Theorem 10 gives the asymptotic slope $\psi(r^* - 1) = \psi(\alpha - \beta - 1)$. Similarly, the left slope is seen to be $\psi(q^*) = \psi(\alpha + \beta)$. We remark that the same slopes were computed in [4] using the tail-wing formula and explicitly known density asymptotics for NIG.

5.3 Criterion II: the Double Exponential Model

The double exponential model $DE = DE(\sigma, \mu, \lambda, p, q, \eta_1, \eta_2)$ has mgf

$$\log M(s) = \frac{1}{2}\sigma^2 s^2 + \mu s + \lambda\left(\frac{p\eta_1}{\eta_1 - s} + \frac{q\eta_2}{\eta_2 + s} - 1\right).$$

Clearly, $r^* = \eta_1$ and as $s \to 0+$

$$\log M(\eta_1 - s) \sim \frac{1}{2}\sigma\eta_1^2 + \mu\eta_1 + \lambda\left(\frac{p\eta_1}{s} + \frac{q\eta_2}{\eta_2 + \eta_1} - 1\right) \sim \lambda p\eta_1 s^{-1}$$

and we see that Criterion II is satisfied. As above, this implies asympotic slopes $\psi(r^* - 1) = \psi(\eta_1 - 1)$ on the right and $\psi(\eta_2)$ on the left.

6 Time changed Lévy processes

We now discuss how to apply our results to time changed Lévy processes [11, 12, 6]. To do this, we only need to check that the moment generating function of the marginals of the process, satisfies one of the three criteria.

To this end, consider a Lévy process $L = L(t)$ described through its cumulant generating function (cgf) K_L at time 1, that is,

$$K_L(v) = \log E[\exp(vL_1)]$$

and an independent random clock $T = T(\omega) \geq 0$ with cgf K_T. It follows that the mgf of $L \circ T$ is given by

$$M(v) = \mathbb{E}\left[\mathbb{E}\left(e^{vL_T}|T\right)\right] = \mathbb{E}\left[e^{K_L(v)T}\right] = \exp\left[K_T(K_L(v)\right].$$

Therefore, in order to apply our Theorem 10 to time-changed Lévy models we need to check if $M = \exp[K_T(K_L(\cdot)]$ satisfies criterion I or II so that $-\log \bar{F}(x)/x$ tends to a positive constant. Here, as earlier, F denotes the distribution function of M and $\bar{F} \equiv 1 - F$. The following theorem gives sufficient conditions for this in terms of K_T and K_L. We shall write $M_T \equiv \exp(K_T)$ and $M_L \equiv \exp(K_L)$. For brevity, we only discuss the right tail[6] and set

$$p_L = \sup\{s : M_L(s) < \infty\}, \;\; p_T = \sup\{s : M_T(s) < \infty\}.$$

[6]In fact, the elegant change-of-measure argument in Lee [9] allows a formal reduction of the left tail behaviour to the right tail behaviour.

Theorem 11 *With notation as above, assuming $p_L, p_T > 0$, we have:*
(i.1) If $K_L(p) = p_T$ for some $p \in [0, p_L)$ and M_T satisfies either criterion then

$$\log \bar{F}(x) \sim -px.$$

(i.2) If $K_L(p) = p_T$ for $p = p_L$ and M_T, M_L satisfy either criterion then

$$\log \bar{F}(x) \sim -px.$$

(ii) If $K_L(p) < p_T$ for all $p \in [0, p_L]$ and M_L satisfies either criterion then

$$\log \bar{F}(x) \sim -p_L x.$$

Remark 12 *It is worth noting that there cannot be more than one solution to $K_L(p) = p_T$. To see this, take any v such that $v > 0$ and $K_L(v) > 0$ (any solution to $K_L(p) = p_T > 0$ will satisfy this!) From $M_L \equiv \exp K_L$ it follows that $M_L(0) = 1$ and $M_L(v) > 1$. By convexity of $M_L(\cdot)$ it is easy to see that $M_L'(v)$ is strictly positive and the same is true for $K_L'(v) = M_L'(v)/M_L(v)$. It follows that $w \geq v \implies K_L(w) \geq K_L(v) > 0$ and the set of all $v > 0 : K_L(v) > 0$ is connected and K_L restricted to this set is strictly increasing. This shows that there is at most one solution to $K_L(p) = p_T$.*

Proof. (i.1) Noting that $p > 0$ let as first assume that M_T satisfies criterion I (at $K_L(p) = p_T$ with some $n \geq 0$) so that for some $\rho > 0$ and $l \in R_0$,

$$M_T^{(n)}(u) \sim (p_T - u)^{-\rho}\, l\left((p_T - u)^{-1}\right) \text{ as } u \uparrow p_T.$$

From $M = M_T \circ K_L$ we have $M' = M_T'(K_L)\, K_L'$ and, by iteration, $M^{(n)}$ equals $M_T^{(n)}(K_L)(K_L')^n$ plus a polynomial in $M_T(\cdot), ..., M_T^{(n-1)}(\cdot)$ which remains bounded when the argument approaches p_T. Noting that $K_L'(p) > 0$ (see remark above) we absorb the factor $[K_L'(p)]^n$ into the slowly varying function and see that

$$M^{(n)}(v) \sim (p_T - K_L(v))^{-\rho} l((p_T - K_L(v))^{-1}) \text{ for } \rho \text{ as above and some } l \in R_0$$

as $K_L(v)$ tends to p_T which follows from $v \uparrow p$. Using analyticity of K_L in $(0, p_L)$ and $K_L'(p) \neq 0$ it is clear that $p_T - K_L(v) \sim K_L'(p)(p - v)$ as $v \uparrow p$ and so

$$M^{(n)}(p - v) \sim K_L'(p)^{-\rho} v^{-\rho} l(1/v) \text{ as } v \to 0+.$$

This shows that M satisfies Criterion I (with the same n as M_T). A similar argument shows that M satisfies Criterion II if M_T does. Either way, the assert tail behaviour of $\log \bar{F}$ follows.
(i.2) The (unlikely!) case $K_L(p_L) = p_T$ involves similar ideas and is left to the reader.
(ii) We now assume that

$$\sup_{p \in [0, p_L]} K_L(p) < p_T < \infty.$$

and M_L satisfies either criterion (at p_L). Since $M_L = \exp K_L$ stays bounded as its argument approaches the critical value p_L it is clear that M_L cannot satisfy criterion II or criterion I with $n = 0$ and there must exist a smallest integer n such that

$$M_L^{(n)}(p_L - x) \sim x^{-\rho} l(x) \quad \text{as } x \to p_L$$

for some $\rho > 0$ and $l \in R_0$. We note that

$$M^{(n)}(v) = (K_L^{(n)}(v) K_T'(K_L(v)) + f(v)) \exp(K_T(K_L(v)))$$

where $f(v)$ is a polynomial function of the first $(n-1)$ derivatives of K_L and the first n derivatives of K_T evaluated at $K_L(v)$, which are all bounded for $0 \leq v \leq p_L$. Noting that positivity of T implies $M_T' > 0$ and hence $K_T' > 0$ we see that as $v \uparrow p_L$

$$M^{(n)}(v) \sim K_L^{(n)}(v) K_T'(K_L(p_L)) M(p_L)\,.$$

Applying this to $K_T(x) \equiv x$ leads immediately to

$$K_L^{(n)}(v) \sim M_L^{(n)}(v)/M_L(v) \sim x^{-\rho} l(x)/M_L(p_L).$$

as $v \uparrow p_L$, and so M satisfies criterion I.

■

We now discuss examples to which the above analysis is applicable. For all examples we plot the total variance smile[7] for several maturities and compare with straight lines[8] of correct slope as predicted by Theorem 10. All plots are based on the calibrations obtained in [12]. This is also where the reader can find more details about the respective model parameters.

6.1 Variance Gamma with Gamma-OU time change

We will consider the Variance Gamma process with a Gamma-Ornstein-Uhlenbeck time change and refer to [12] for details. From earlier, the Variance Gamma process has cumulant generating function

$$K_L(v) = C \log\left(\frac{gm}{(m-v)(v+g)}\right) \text{ for } v \in (-g, m)$$

We note that $K_L([0,m]) = [0,\infty]$ so that $p_L = \infty$. The Gamma-Ornstein-Uhlenbeck clock $T = T(\omega, t)$ has cgf

$$K_T(v) = v y_0 \lambda^{-1}(1 - e^{-\lambda t}) + \frac{\lambda a}{v - \lambda b}\left[b \log\left(\frac{b}{b - v\lambda^{-1}(1 - e^{-\lambda t})}\right) - vt\right]$$

We need to examine how this function behaves around the endpoint of its strip of regularity. At first glance, it appears that the function tends to infinity as

[7] That is, $V^2(k,t) \equiv \sigma^2(k,t)\, t$.

[8] These lines have been parallel-shifted so that they are easier to compare with the actual smile.

$v \uparrow \lambda b$, because of the $\frac{\lambda a}{v-\lambda b}$ term. However, upon closer examination, we can see that this is in fact a removable singularity, and the term of interest to us is the log(...) term. This term tends to infinity as $v \to \lambda b(1-e^{-\lambda t})^{-1} =: p_T$. After some simple algebra, we see that

$$\begin{aligned} e^{K_T(v)} &= \left(\frac{b}{b - v\lambda^{-1}(1-e^{-\lambda t})}\right)^{\frac{\lambda ab}{v-\lambda b}} \exp\left\{ v y_0 \lambda^{-1}(1-e^{-\lambda t}) - \frac{vt\lambda a}{v-\lambda b}\right\} \\ &\sim \left(\frac{p_T}{p_T - v}\right)^{\frac{\lambda ab}{p_T-\lambda b}} \exp\left\{ p_T y_0 \lambda^{-1}(1-e^{-\lambda t}) - \frac{p_T t\lambda a}{p_T-\lambda b}\right\} \text{ as } v \uparrow p_T. \end{aligned}$$

Therefore, $\exp(K_T)$ satisfies Criterion I with $n = 0$ and part (i.1) of Theorem 11 shows that M does too and so that $\log \bar{F}(x) \sim -px$ where p is determined by the equation

$$K_L(p) = p_T = \lambda b(1-e^{-\lambda t})^{-1}$$

and can be calculated explicitly,

$$p = \frac{m - g + \sqrt{(m-g)^2 + 4gm(1-\exp(-\lambda b/C(1-e^{-\lambda t}))}}{2}.$$

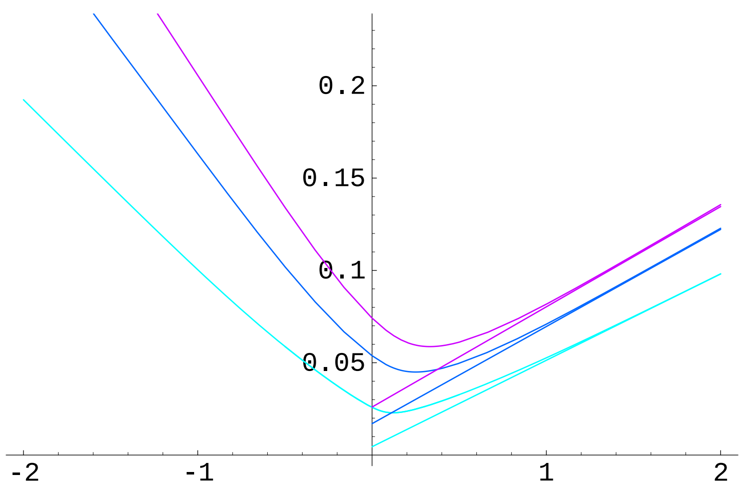

Figure 1: VG with Gamma-OU time change. Parameters from [12]. Total implied variance and slopes for three maturities $t = 0.4, 0.9$ and 1.3 years.

6.2 Normal Inverse Gaussian with CIR time change

The cgf of the Cox-Ingersoll-Ross (CIR) clock $T = T(\omega, t)$ is given by

$$K_T(v) = \kappa^2\eta t/\lambda^2 + 2y_0 v/(\kappa + \gamma \coth(\gamma t/2)) - \frac{2\kappa\eta}{\lambda^2}\log[\sinh\gamma t/2(\coth\gamma t/2 + \frac{\kappa}{\gamma})]$$

where

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 v}.$$

This clearly tends to infinity as $I(v) \equiv \kappa + \gamma(v)\coth(\gamma(v)t/2) \to 0$, and we can define p_T as solution to the equation $I(p_T) = 0$. Using l'Hôpital's rule, it is easy to check that

$$\frac{p_T - v}{\kappa + \gamma(v)\coth(\gamma(v)t/2)} t$$

tends to a constant as $v \to p_T$, and so $2y_0 v/(\kappa+\gamma\coth(\gamma t/2)$ is regularly varying of index 1 as a function of $(p_T - v)^{-1}$. It is clear that this is the dominant term in this limit, and so $M_T \equiv \exp(K_T)$ satisfies criterion II (at p_T). From earlier, the NIG cgf is[9]

$$K_L(v) = -\delta(\sqrt{\alpha^2 - (\beta + v)^2} - \sqrt{\alpha^2 - \beta^2}) \text{ for } v \leq \alpha - \beta$$

from which we see that $p_L = \alpha - \beta > 0$ and

$$\sup_{v \in [0,\alpha-\beta]} K_L(v) = \delta\sqrt{\alpha^2 - \beta^2}.$$

Therefore, the behavior of M on the edge of the strip of analyticity, and the location of the critical value, will depend on whether this supremum is more or less than p_T; if it is less than p_T, the latter is never reached. Recalling that $\exp(K_L)$ satisfies Criterion I with $n = 1$, we apply part (ii) of Theorem 11 and obtain

$$-\log \bar{F}(x) \sim p_L x = (\alpha - \beta)x.$$

Otherwise, there exists $p \in (0, \alpha - \beta]$ such that $K_L(p) = p_T$, for some $p \leq \alpha - \beta$, and since M_T was seen to satisfy one of the criteria (to be precise: Criterion II) we can apply part (i) of Theorem 11 and obtain

$$-\log \bar{F}(x) \sim px.$$

In particular, we see that for all possible parameters in the NIG-CIR model the formula (2) holds true. Smile-asymptotics are now an immediate consequence from Theorem 9.

6.3 The Heston Model

The Heston model is a stochastic volatility model defined by the following stochastic differential equations:

$$\frac{dS_t}{S_t} = \sqrt{v_t} dW_t^1$$
$$dv_t = \kappa(\eta - v_t)dt + v_t dW_t^2$$

[9] Following [12] we take $\mu = 0$ here.

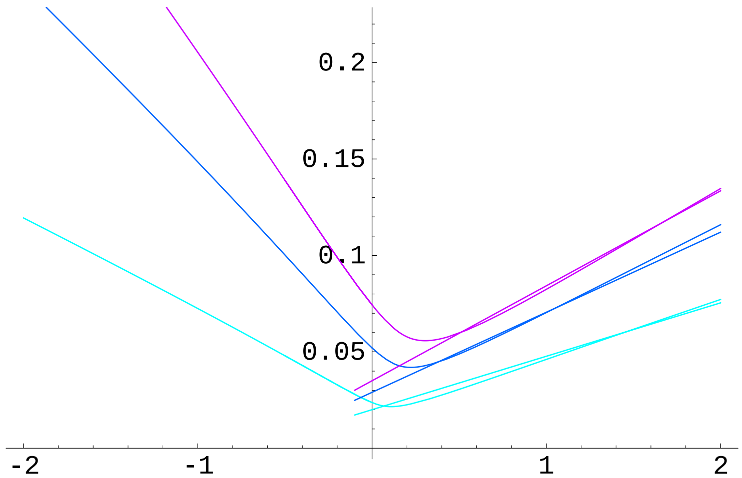

Figure 2: NIG with CIR time change. Parameters from [12]. Total implied variance and slopes for three maturities $t = 0.4, 0.9$ and 1.3 years. Observe that the the lines with correct slope do not perfectly line up with the smile which is *not* a contradiction to the result that that $V^2(k)/k$ converges to a constant.

where $d\left\langle W_t^1, W_t^2 \right\rangle = \rho dt$ is the correlation of the two Brownian motions. $\log S_t$ therefore has the distribution of a Brownian motion with drift $-1/2$ evaluated at a random time $T(\omega, t) = \int_0^t v_s ds$ with the distribution of an integrated CIR process, as in the previous example. When $\rho = 0$, the Lévy process $L \equiv W^1$ and T are independent and we can apply the same analysis as above. Namely, the cgf of the Brownian motion with drift speed $-1/2$ at time 1 is

$$K_L(v) = (v^2 - v)/2,$$

so that $p_L = \infty$, and $M_T = \exp(K_T)$ satisfies Criterion II hence, by part (i) of Theorem 11,

$$\log \bar{F}(x) \sim -px$$

where p is determined by the equation $K_L(p) = p_T$. When $\rho \leq 0$, we can analyze the mgf of $\log S_t$ directly, and we can apply the same reasoning as for the mgf of the CIR process, to deduce that criterion II is satisfied. The distribution function for the Heston returns hence satisfies $\log \bar{F}(x) \sim -px$ where p is solution to, see [3],

$$(\kappa - \rho v\theta) + (\theta^2(v^2 - v) - (\kappa - \rho v\theta)^2)^{1/2} \cot\{(\theta^2(v^2 - v) - (\kappa - \rho v\theta)^2)^{1/2} t/2\}\Big|_{v=p} = 0.$$

When $\rho > 0$, which is of little practical importance (at least in equity markets), the mgf may explode at a different point, see [3], but criterion II will still be satisfied.

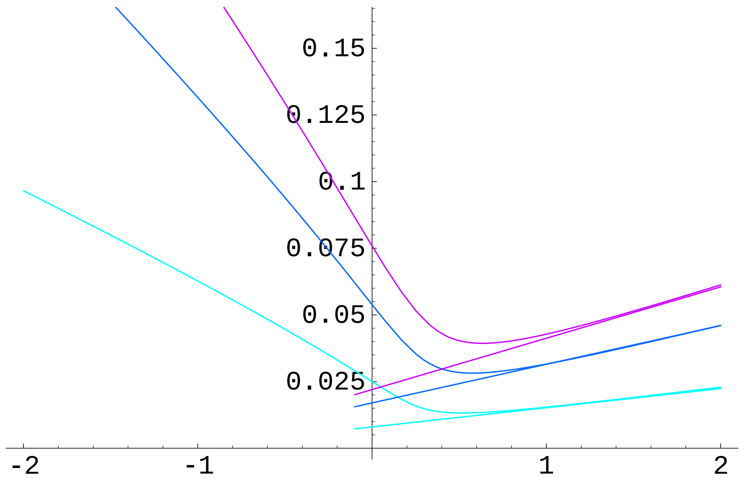

Figure 3: Heston Model. Parameters from [12]. Total implied variance and slopes for three maturities $t = 0.4, 0.9$ and 1.3 years.

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